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Research of the periodic solutions of the Hénon–Heiles nonintegrable Hamiltonian system by the Lindstedt–Poincaré method

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Abstract. In this paper, we seek the periodic solutions of the Hénon–Heiles nonintegrable Hamiltonian system. We apply the Lindstedt–Poincaré method, in order, first to enumerate the main periodic families in the neighbourhood of the origin, then to determine the series corresponding to these families and to their periods. All the series will be computed to $O(A^{21})$ by means of the computer algebra system 'Mathematica', where A is the zeroth-order amplitude. We also prove that the period of the rectilinear periodic family is exactly equal to a Gauss hypergeometric series. Moreover, we show that the celestial technique of the 'elimination of secular terms' is rigorously equivalent to the 'Fredholm alternative'. We further test the validity of the periodic families using numerical integration. Finally, we compare our results with those of the Churchill–Pecelli–Rod 'geometrical' method.

1. Introduction

In this work, we will apply the Lindstedt–Poincaré (LP) method to look for the periodic solutions, in the neighbourhood of the equilibrium point (here the origin), of the Hénon–Heiles nonintegrable Hamiltonian system whose Hamiltonian is given by

$$H = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + x^2 + y^2) + \epsilon(xy^2 - \tfrac{1}{3}x^3) \tag{1.1}$$

where ϵ is a real parameter and (x, y) are the generalized coordinates with $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$.

Using the technique of the stretching variables $\tilde{x} = \epsilon x$, $\tilde{y} = \epsilon y$, $\tilde{H} = \epsilon^2 H$, we can set $\epsilon = 1$. The motion of this system is governed by the following differential equations:

$$\begin{aligned} \ddot{x} + x &= \epsilon(x^2 - y^2) \\ \ddot{y} + y &= -2\epsilon xy. \end{aligned} \tag{1.2}$$

In view of the theorem of Weinstein–Moser [1], we can assure the existence for this system, for sufficiently small energies, at least two periodic orbits.

Recently, this system aroused the increasing interest of many astronomers, physicists and mathematicians [2–4]. Initially, it was introduced and studied numerically by the two astronomers Hénon and Heiles [5]. Later, it was the subject of numerous numerical and geometrical researches. One of the major problems met with the application of numerical methods to Hamiltonian systems is the accumulation of round-off errors. These last methods also present the inconvenience that they do not offer any information about the number of the periodic families nor about their periods.

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In our work, we shall be interested in the Hénon–Heiles system from a perturbative point of view, using the LP method. We will see that the importance of this method lies in the fact that it permits on the one hand the enumeration of the main periodic families and on the other hand the determination of these families, as well as their periods in the form of perturbative series.

This method is well known in the literature of one-degree-of-freedom systems [6, 7]. It remains very little used in the case of two-degrees-of-freedom systems, in spite of the existence of powerful computer algebra systems. The power of these systems is due to the fact that they handle not only symbolic computations but also numerical computations with any precision.

In section 2, we will describe the LP method in the case of the Hénon–Heiles system, thus deducing a recurrent functional algorithm. We will also prove that the celestial technique of 'elimination of secular terms', on which the LP method is based, is rigorously equivalent to the 'alternative of Fredholm'.

In section 3, we will apply the LP method in order to look for the periodic families of the Hénon–Heiles system in the neighbourhood of the origin. We will then show, exploiting the first steps of the LP algorithm, that the system has eight main periodic families. Actually, taking into account the symmetry and the invariance by rotations of angles $\pm\frac{2\pi}{3}$ of the potential, we can distinguish only three main periodic families: the 'rectilinear-$\mathcal{R}$', the 'curvilinear-$\mathcal{V}$' and the 'circular-$\mathcal{C}$'. By means of the computer algebra system 'Mathematica', we will compute the series corresponding to these periodic families and to their periods to $\mathrm{O}(A^{21})$, where A is the zeroth-order amplitude. We will moreover test the validity of these series by numerical integration.

In section 4, we will study the periods of the three main periodic families in terms of the energy E. We will first give the coefficients of the power series, truncated to $\mathrm{O}(E^{10})$, representing the periods of these families. We will then prove, reducing the system to one degree of freedom and applying a method based on the distributions due to the authors of [8, 9], that the period of the rectilinear family is equal to a Gauss hypergeometric series. This will constitute a good check of the rectilinear periodic family computed by the LP method. We will finally discuss the convergence of the series representing the periods of the three main periodic families.

In section 5, we will compare our results with those of the 'geometrical' method of Churchill *et al* [10]. We will particularly notice the perfect agreement concerning the number and the form of the main periodic families. However, we will point out a disagreement about the circular periodic family.

2. Method of Lindstedt–Poincaré

2.1. Description of the Lindstedt–Poincaré method

The main purpose of this method is to look for periodic solutions of the system (1.2) in the form

$$\begin{aligned} x(t) &= \sum_{j=0}^{\infty} x_j(t)\cdot\epsilon^j \\ y(t) &= \sum_{j=0}^{\infty} y_j(t)\cdot\epsilon^j \end{aligned} \qquad (2.1)$$

where the functions x_j and y_j are T-periodic. Moreover, the method requires that the pulsation $\omega = \frac{2\pi}{T}$ is in the form

$$\omega = \omega_0 + \omega_1 \cdot \epsilon + \omega_2 \cdot \epsilon^2 + \cdots \tag{2.2}$$

where $\omega_0 = 1$ and $\omega_j \in \mathbb{R}$.

Let us make the change of variables

$$\begin{aligned} &\theta = \omega t \\ &X(\theta) = x(t) \qquad X_j(\theta) = x_j(t) \\ &Y(\theta) = y(t) \qquad Y_j(\theta) = y_j(t). \end{aligned} \tag{2.3}$$

The system (1.2) then becomes

$$\begin{aligned} &\omega^2 X'' + X = \epsilon(X^2 - Y^2) \\ &\omega^2 Y'' + Y = -2\epsilon XY. \end{aligned} \tag{2.4}$$

The problem is now reduced to looking for the 2π-periodic solutions (X, Y) in the form

$$\begin{aligned} &X(\theta) = \sum_{j=0}^{\infty} X_j(\theta) \cdot \epsilon^j \\ &Y(\theta) = \sum_{j=0}^{\infty} Y_j(\theta) \cdot \epsilon^j \end{aligned} \tag{2.5}$$

where the functions X_j and Y_j are 2π-periodic.

Setting

$$\begin{aligned} &F(\theta) = \epsilon(X^2(\theta) - Y^2(\theta)) \\ &G(\theta) = -2\epsilon X(\theta)Y(\theta) \end{aligned} \tag{2.6}$$

we obtain

$$\begin{aligned} &F(\theta) = \sum_{j=0}^{\infty} F_j(\theta) \cdot \epsilon^j \\ &G(\theta) = \sum_{j=0}^{\infty} G_j(\theta) \cdot \epsilon^j \end{aligned} \tag{2.7}$$

with

$$\begin{aligned} &F_0 = G_0 = 0 \\ &F_j = \sum_{k=0}^{j-1} (X_k \cdot X_{j-k-1} - Y_k \cdot Y_{j-k-1}) \qquad j \geqslant 1 \\ &G_j = -2 \sum_{k=0}^{j-1} X_k \cdot Y_{j-k-1} \qquad j \geqslant 1. \end{aligned} \tag{2.8}$$

The series ω^2 can be written in the form

$$\begin{aligned} &\omega^2 = \sum_{j=0}^{\infty} Q_j \cdot \epsilon^j \\ &Q_j = \sum_{k=0}^{j} \omega_k \cdot \omega_{j-k}. \end{aligned} \tag{2.9}$$

Equating the coefficients of ϵ^j on both sides of the system (2.4) we obtain the algorithm

$$\sum_{k=0}^{j} Q_{j-k} \begin{pmatrix} X_k'' \\ Y_k'' \end{pmatrix} + \begin{pmatrix} X_j \\ Y_j \end{pmatrix} = \begin{pmatrix} F_j \\ G_j \end{pmatrix} \qquad j \in \mathbb{N} \tag{2.10}$$

which can be written in the form

$$\begin{pmatrix} X_j'' + X_j \\ Y_j'' + Y_j \end{pmatrix} = \begin{pmatrix} F_j \\ G_j \end{pmatrix} - \sum_{k=1}^{j-1} Q_{j-k} \begin{pmatrix} X_k'' \\ Y_k'' \end{pmatrix} - Q_j \begin{pmatrix} X_0'' \\ Y_0'' \end{pmatrix} \qquad j \in \mathbb{N}. \tag{2.11}$$

This sequence of second-order linear differential systems of unknowns (X_j, Y_j), constitutes a recurrent functional algorithm permitting in each step $j \in \mathbb{N}$ to simultaneously determine the constant Q_j and the solution (X_j, Y_j).

Determination of Q_j *and* (X_j, Y_j). We are now going to show how to determine recurrently Q_j and (X_j, Y_j).

Step $j = 0$. The system of this step is

$$\begin{aligned} X_0'' + X_0 &= 0 \\ Y_0'' + Y_0 &= 0 \end{aligned} \tag{2.12}$$

whose solution is given by

$$\begin{aligned} X_0 &= A\cos\theta + B\sin\theta \\ Y_0 &= C\cos\theta + D\sin\theta \end{aligned} \tag{2.13}$$

where A, B, C and D are arbitrary constants of integration.

We also have here

$$Q_0 = 1. \tag{2.14}$$

Step $j = 1$. We have here the system

$$\begin{pmatrix} X_1'' + X_1 \\ Y_1'' + Y_1 \end{pmatrix} = \begin{pmatrix} F_1 \\ G_1 \end{pmatrix} - Q_1 \begin{pmatrix} X_0'' \\ Y_0'' \end{pmatrix} \tag{2.15}$$

where

$$\begin{pmatrix} F_1 \\ G_1 \end{pmatrix} = \begin{pmatrix} X_0^2 - Y_0^2 \\ -2X_0Y_0 \end{pmatrix}. \tag{2.16}$$

After some calculations we obtain

$$\begin{pmatrix} F_1 \\ G_1 \end{pmatrix}(\theta) = \begin{pmatrix} a_{10} \\ c_{10} \end{pmatrix} + \sum_{k=1}^{2} \begin{pmatrix} a_{1k} & b_{1k} \\ c_{1k} & d_{1k} \end{pmatrix} \begin{pmatrix} \cos k\theta \\ \sin k\theta \end{pmatrix} \tag{2.17}$$

with

$$\begin{aligned} & a_{10} = \frac{A^2 + B^2 - C^2 - D^2}{2} \qquad c_{10} = -AC - BD \\ & a_{11} = 0 \qquad c_{11} = 0 \\ & a_{12} = \frac{A^2 + D^2 - B^2 - C^2}{2} \qquad c_{12} = BD - AC \\ & b_{11} = 0 \qquad d_{11} = 0 \\ & b_{12} = AB - CD \qquad d_{12} = -AD - BC. \end{aligned} \tag{2.18}$$

The system (2.15) then becomes

$$\begin{pmatrix} X_1'' + X_1 \\ Y_1'' + Y_1 \end{pmatrix} = \begin{pmatrix} a_{10} \\ c_{10} \end{pmatrix} + \begin{pmatrix} a_{11} + A \cdot Q_1 & b_{11} + B \cdot Q_1 \\ c_{11} + C \cdot Q_1 & d_{11} + D \cdot Q_1 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \begin{pmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{pmatrix} \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix}. \tag{2.19}$$

Determination of Q_1. In order to avoid the secular terms ($\theta\cos\theta$ and $\theta\sin\theta$) in X_1 and Y_1, the coefficients of $\cos\theta$ and $\sin\theta$ have to vanish in the system (2.19).

So we obtain four algebraic equations

$$\begin{aligned} a_{11} + A \cdot Q_1 &= 0 \qquad c_{11} + C \cdot Q_1 = 0 \\ b_{11} + B \cdot Q_1 &= 0 \qquad d_{11} + D \cdot Q_1 = 0. \end{aligned} \tag{2.20}$$

The constant Q_1 exists then if and only if the six determinants

$$\begin{aligned} D_1(A,B) &= Ba_{11} - Ab_{11} \qquad D_1(B,C) = Cb_{11} - Bc_{11} \\ D_1(A,C) &= Ca_{11} - Ac_{11} \qquad D_1(B,D) = Db_{11} - Bd_{11} \\ D_1(A,D) &= Da_{11} - Ad_{11} \qquad D_1(C,D) - Dc_{11} - Cd_{11} \end{aligned} \tag{2.21}$$

are null. In our case they vanish since we have

$$a_{11} = b_{11} = c_{11} = d_{11} = 0.$$

We can then deduce the constant Q_1 from one of the equations (2.20)

$$Q_1 = -\frac{a_{11}}{A} = 0 \qquad (\text{if } A \neq 0). \tag{2.22}$$

Determination of X_1 *and* Y_1. After eliminating the terms $\cos\theta$ and $\sin\theta$, the resolution of the system (2.19) gives

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} A_1 B_1 \\ C_1 D_1 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \begin{pmatrix} a_{10} \\ c_{10} \end{pmatrix} + \sum_{k=2}^{2} \left(\frac{1}{1-k^2} \right) \begin{pmatrix} a_{1k} & b_{1k} \\ c_{1k} & d_{1k} \end{pmatrix} \begin{pmatrix} \cos k\theta \\ \sin k\theta \end{pmatrix} \tag{2.23}$$

where A_1, B_1, C_1 and D_1 are arbitrary constants of integration. Since we have not yet obtained a relation between A, B, C and D, we can set $A_1 = B_1 = C_1 = D_1 = 0$.

Step j. The system of this step is

$$\begin{pmatrix} X_j'' + X_j \\ Y_j'' + Y_j \end{pmatrix} = \begin{pmatrix} F_j \\ G_j \end{pmatrix} - \sum_{k=1}^{j-1} Q_{j-k} \begin{pmatrix} X_k'' \\ Y_k'' \end{pmatrix} - Q_j \begin{pmatrix} X_0'' \\ Y_0'' \end{pmatrix}. \tag{2.24}$$

Now suppose, by hypothesis of induction, we have determined Q_i and (X_i, Y_i) for all $0 \leqslant i \leqslant j-1$ with

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \begin{pmatrix} a_{i0} \\ c_{i0} \end{pmatrix} + \sum_{k=2}^{i+1} \left(\frac{1}{1-k^2} \right) \begin{pmatrix} a_{ik} & b_{ik} \\ c_{ik} & d_{ik} \end{pmatrix} \begin{pmatrix} \cos k\theta \\ \sin k\theta \end{pmatrix} \tag{2.25}$$

where A_i, B_i, C_i, D_i, are arbitrary constants of integration.

We can then prove easily that at this step j we have

$$\begin{pmatrix} F_j \\ G_j \end{pmatrix} - \sum_{k=1}^{j-1} Q_{j-k} \begin{pmatrix} X_k'' \\ Y_k'' \end{pmatrix} = \begin{pmatrix} a_{j0} \\ c_{j0} \end{pmatrix} + \sum_{k=2}^{j+1} \begin{pmatrix} a_{jk} & b_{jk} \\ c_{jk} & d_{jk} \end{pmatrix} \begin{pmatrix} \cos k\theta \\ \sin k\theta \end{pmatrix} \tag{2.26}$$

where a_{jk}, b_{jk}, c_{jk} and d_{jk} depend on the constants A, B, C and D.

The system (2.24) then becomes

$$\begin{pmatrix} X_j'' + X_j \\ Y_j'' + Y_j \end{pmatrix} = \begin{pmatrix} a_{j0} \\ c_{j0} \end{pmatrix} + \begin{pmatrix} a_{j1} + A \cdot Q_j & b_{j1} + B \cdot Q_j \\ c_{j1} + C \cdot Q_j & d_{j1} + D \cdot Q_j \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \sum_{k=2}^{j+1} \begin{pmatrix} a_{jk} b_{jk} & \\ c_{jk} & d_{jk} \end{pmatrix} \begin{pmatrix} \cos k\theta \\ \sin k\theta \end{pmatrix}. \tag{2.27}$$

Determination of Q_j. Using the same technique of 'elimination of secular terms' applied in the step $j = 1$, we obtain four algebraic equations

$$\begin{aligned} a_{j1} + A \cdot Q_j &= 0 \qquad & c_{j1} + C \cdot Q_j &= 0 \\ b_{j1} + B \cdot Q_j &= 0 \qquad & d_{j1} + D \cdot Q_j &= 0. \end{aligned} \tag{2.28}$$

The constant Q_j exists then if and only if the six determinants

$$\begin{aligned} D_j(A, B) &= Ba_{j1} - Ab_{j1} \qquad & D_j(B, C) &= Cb_{j1} - Bc_{j1} \\ D_j(A, C) &= Ca_{j1} - Ac_{j1} \qquad & D_j(B, D) &= Db_{j1} - Bd_{j1} \\ D_j(A, D) &= Da_{j1} - Ad_{j1} \qquad & D_j(C, D) &= Dc_{j1} - Cd_{j1} \end{aligned} \tag{2.29}$$

are null.

Setting these determinants equal to zero permits us to eventually find a relation between the constants of integration introduced in the preceding steps. To determine Q_j it is sufficient to use one of the equations (2.28)

$$Q_j = -\frac{a_{j1}}{A} (\text{if } A \neq 0). \tag{2.30}$$

Determination of X_j *and* Y_j. After eliminating the terms $\cos(\theta)$ and $\sin(\theta)$, the resolution of the system (2.24) gives

$$\begin{pmatrix} X_j \\ Y_j \end{pmatrix} = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \begin{pmatrix} a_{j0} \\ c_{j0} \end{pmatrix} + \sum_{k=2}^{j+1} \left(\frac{1}{1 - k^2} \right) \begin{pmatrix} a_{jk} & b_{jk} \\ c_{jk} & d_{jk} \end{pmatrix} \begin{pmatrix} \cos k\theta \\ \sin k\theta \end{pmatrix} \tag{2.31}$$

where A_j, B_j, C_j and D_j are arbitrary constants of integration.

2.2. Fredholm alternative and secular terms

We will now show that the well known celestial technique of 'elimination of secular terms', on which the LP method is based, is rigorously equivalent to the Fredholm alternative.

Let us now rewrite the linear differential system (2.24) in the form

$$\begin{pmatrix} X_j'' \\ Y_j'' \end{pmatrix} + \begin{pmatrix} X_j \\ Y_j \end{pmatrix} = \begin{pmatrix} f_j \\ g_j \end{pmatrix} \tag{2.32}$$

where

$$\begin{pmatrix} f_j \\ g_j \end{pmatrix} = \begin{pmatrix} a_{j0} \\ c_{j0} \end{pmatrix} + \begin{pmatrix} a_{j1} + AQ_j \\ c_{j1} + CQ_j \end{pmatrix} \cos\theta + \begin{pmatrix} b_{j1} + BQ_j \\ d_{j1} + DQ_j \end{pmatrix} \sin\theta + \sum_{k=2}^{j+1} \begin{pmatrix} a_{jk} & b_{jk} \\ c_{jk} & d_{jk} \end{pmatrix} \begin{pmatrix} \cos k\theta \\ \sin k\theta \end{pmatrix}. \tag{2.33}$$

Theorem. (Fredholm alternative, cf Reinhard [11, p 394]). Let us consider the differential equation in $\mathbb{R}^d$

$$\text{(I)} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + b(t)$$

where the functions b and A, defined in $\mathbb{R}$ and with values respectively in $\mathbb{R}^d$ and $\mathcal{L}(\mathbb{R}^d)$, are continuous and T-periodic. We assume that the following adjoint equation of the homogeneous equation

$$\text{(II)} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\,{}^{t}A(t) \cdot y$$

admits p independent T-periodic solutions $y_1, y_2, \ldots, y_p$. Then there exists p independent T-periodic solutions for the equation (I) if and only if for all $k \in \mathbb{N}$ such that $k \leqslant p$ we have

$$\text{(III)} \qquad \int_0^T \langle y_k(t), b(t) \rangle \, \mathrm{d}t = 0$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$.

Let us apply this theorem to the system (2.32). If we set

$$\begin{aligned} u_1 &= X_j \\ u_2 &= X_j' \\ u_3 &= Y_j \\ u_4 &= Y_j' \end{aligned} \tag{2.34}$$

then the system (2.32) can be written in the normal form

$$\begin{pmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} + \begin{pmatrix} 0 \\ f_j \\ 0 \\ g_j \end{pmatrix} \tag{2.35}$$

or equivalently in the condensed form

$$u' = Au + b. \tag{2.36}$$

It is clear that A and b are 2π-periodic. The adjoint equation of (2.36) is then given by

$$v' = Av. \tag{2.37}$$

We can easily prove that this equation admits the four following independent 2π-periodic solutions

$$v_1 = \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ \cos\theta \\ -\sin\theta \end{pmatrix} \qquad v_4 = \begin{pmatrix} 0 \\ 0 \\ \sin\theta \\ \cos\theta \end{pmatrix} \tag{2.38}$$

which are the rows of the resolvant $\mathrm{e}^{A\theta}$.

In view of the preceding theorem, the system (2.36) admits 2π-periodic solutions if and only if for all $k \in \mathbb{N}$ such that $1 \leqslant k \leqslant 4$ we have

$$\int_0^{2\pi} \langle v_k(\theta), b(\theta) \rangle \, \mathrm{d}\theta = 0 \tag{2.39}$$

i.e.

$$\begin{aligned}\int_0^{2\pi}\begin{pmatrix} f_j(\theta)\\ g_j(\theta)\end{pmatrix}\cos\theta\,\mathrm{d}\theta &= \begin{pmatrix}0\\0\end{pmatrix}\\ \int_0^{2\pi}\begin{pmatrix} f_j(\theta)\\ g_j(\theta)\end{pmatrix}\sin\theta\,\mathrm{d}\theta &= \begin{pmatrix}0\\0\end{pmatrix}\end{aligned}. \tag{2.40}$$

The first equality is equivalent to

$$\begin{aligned}&\begin{pmatrix}a_{j0}\\ c_{j0}\end{pmatrix}\int_0^{2\pi}\cos\theta\,\mathrm{d}\theta + \begin{pmatrix}a_{j1}+AQ_j\\ c_{j1}+CQ_j\end{pmatrix}\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta + \begin{pmatrix}b_{j1}+BQ_j\\ d_{j1}+DQ_j\end{pmatrix}\int_0^{2\pi}\sin\theta\cdot\cos\theta\,\mathrm{d}\theta\\ &\quad+\sum_{k=2}^{j+1}\begin{pmatrix}a_{jk} & b_{jk}\\ c_{jk} & d_{jk}\end{pmatrix}\begin{pmatrix}\int_0^{2\pi}\cos k\theta\cos\theta\,\mathrm{d}\theta\\ \int_0^{2\pi}\sin k\theta\cos\theta\,\mathrm{d}\theta\end{pmatrix}\\ &= \begin{pmatrix}0\\0\end{pmatrix}.\end{aligned} \tag{2.41}$$

It is evident that all the integrals in the relation (2.41) are null except $\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta = \pi$. The relation (2.41) is therefore equivalent to two algebraic equations

$$\begin{aligned}a_{j1}+AQ_j &= 0\\ c_{j1}+CQ_j &= 0.\end{aligned} \tag{2.42}$$

Proceeding in the same manner for the second equality of (2.40), we obtain two other algebraic equations

$$\begin{aligned}b_{j1}+BQ_j &= 0\\ d_{j1}+DQ_j &= 0.\end{aligned} \tag{2.43}$$

We thus recover the four algebraic equations (2.28) obtained using the technique of ‘elimination of secular terms’.

3. Enumeration and computation of the periodic families of the Hénon–Heiles system

3.1. Enumeration of the main periodic families of the Hénon–Heiles system

We will see here that the enumeration of the main periodic families is accomplished at the fourth step of the algorithm (2.11).

Step $j = 0$. The solution of this step is

$$\begin{aligned}X_0 &= A\cos\theta + B\sin\theta\\ Y_0 &= C\cos\theta + D\sin\theta\\ Q_0 &= 1.\end{aligned} \tag{3.1}$$

Step $j = 1$. The solution here is

$$\begin{aligned}X_1 &= a_{10} - \tfrac{1}{3}(a_{12}\cos 2\theta + b_{12}\sin 2\theta\\ Y_1 &= c_{10} - \tfrac{1}{3}(c_{12}\cos 2\theta + d_{12}\sin 2\theta)\\ Q_1 &= 0\end{aligned} \tag{3.2}$$

where the coefficients a_{1j}, b_{1j}, c_{1j} and d_{1j} are given in (2.18).

Step $j = 2$. To compute the six determinants D_2 of this step we must calculate the coefficients a_{21}, b_{21}, c_{21} and d_{21}

$$
\begin{aligned}
a_{21} &= \tfrac{5}{6}(A^3 + AB^2 + AC^2) + \tfrac{7}{3}BCD - \tfrac{3}{2}AD^2 \\
b_{21} &= \tfrac{5}{6}(B^3 + BA^2 + BD^2) + \tfrac{7}{3}ACD - \tfrac{3}{2}BC^2 \\
c_{21} &= \tfrac{5}{6}(C^3 + CD^2 + CA^2) + \tfrac{7}{3}ABD - \tfrac{3}{2}CB^2 \\
d_{21} &= \tfrac{5}{6}(D^3 + DC^2 + DB^2) + \tfrac{7}{3}ABC - \tfrac{3}{2}DA^2.
\end{aligned}
\tag{3.3}
$$

We then deduce the six determinants

$$
\begin{aligned}
D_2(A, B) &= \tfrac{7}{3}(AC + BD)(BC - AD) \qquad & D_2(B, C) &= \tfrac{7}{3}(B^2 - C^2)BC - AD) \\
D_2(A, C) &= \tfrac{7}{3}(AB + CD)(BC - AD) \qquad & D_2(B, D) &= \tfrac{7}{3}(AB + CD)(AD - BC) \\
D_2(A, D) &= \tfrac{7}{3}(A^2 - D^2)(AD - BC) \qquad & D_2(C, D) &= \tfrac{7}{3}(AC + BD)(AD - BC)
\end{aligned}
\tag{3.4}
$$

which have to vanish. We can distinguish two cases.

$$
\begin{array}{llcl}
\text{First case} & (A, B) = (0, 0) & \text{or} & (C, D) = (0, 0) \\
\text{Second case} & (A, B) \neq (0, 0) & \text{and} & (C, D) \neq (0, 0).
\end{array}
$$

The zeroth-order solution given in (3.1) may be written as

$$
\begin{aligned}
X_0 &= \alpha_1 \cos(\theta + \theta_1) \\
Y_0 &= \alpha_2 \cos(\theta + \theta_2).
\end{aligned}
\tag{3.5}
$$

First case. $(A, B) = (0, 0)$ or $(C, D) = (0, 0)$

(i) $(A, B) = (0, 0)$. Thus

$$
\begin{aligned}
X_0 &= 0 \\
Y_0 &= \alpha_2 \cos(\theta + \theta_2).
\end{aligned}
$$

As the system is autonomous, we can make the change of variable $\theta \rightarrow \theta - \theta_2$. We then obtain

$$
\begin{aligned}
X_0 &= 0 \\
Y_0 &= C \cos\theta.
\end{aligned}
\tag{3.6}
$$

In this case all six determinants (3.4) are null without any condition.

(ii) $(C, D) = (0, 0$. Thus

$$
\begin{aligned}
X_0 &= \alpha_1 \cos(\theta + \theta_1) \\
Y_0 &= 0
\end{aligned}
$$

which may be written, after the change of variable $\theta \rightarrow \theta - \theta_1$, in the form

$$
\begin{aligned}
X_0 &= A \cos\theta \\
Y_0 &= 0.
\end{aligned}
\tag{3.7}
$$

In this case we also remark that the six determinants (3.4) are all null without any condition.

Second case. $(A, B) \neq (0, 0)$ and $(C, D) \neq (0, 0)$. In this case we have

$$\begin{aligned} &\alpha_1 = \sqrt{A^2 + B^2} && \alpha_2 = \sqrt{C^2 + D^2} \\ &\sin\theta_1 = -\frac{B}{\alpha_1} && \sin\theta_2 = -\frac{D}{\alpha_2} \\ &\cos\theta_1 = \frac{A}{\alpha_1} && \cos\theta_2 = \frac{C}{\alpha_2}. \end{aligned} \tag{3.8}$$

If we equate the determinant $D_2(A, B)$ given in (3.4) to zero, we obtain

$$BC - AD = 0 \qquad \text{or} \qquad AC + BD = 0 \tag{3.9}$$

which is equivalent to

$$\begin{aligned} &\sin(\theta_2 - \theta_1) = 0 && \text{or} && \cos(\theta_2 - \theta_1) = 0 \\ &\Leftrightarrow \theta_2 - \theta_1 = k\pi && \text{or} && \theta_2 - \theta_1 = \frac{\pi}{2} + k\pi. \end{aligned} \tag{3.10}$$

Hence $(\theta_2 - \theta_1) \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$.

As the system is autonomous, we can make the change of variable $\theta \to \theta - \theta_1$. We thus have

$$\begin{aligned} &X_0 = \alpha_1 \cos\theta \\ &Y_0 = \alpha_2 \cos(\theta + \theta_2 - \theta_1). \end{aligned} \tag{3.11}$$

Since we have $(\theta_2 - \theta_1) \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, we can distinguish two cases

$$\begin{aligned} &X_0 = A\cos\theta && \text{and} && X_0 = A\cos\theta \\ &Y_0 = C\cos\theta && \text{and} && Y_0 = D\sin\theta. \end{aligned} \tag{3.12}$$

It is clear that the previous cases (3.6) and (3.7) are included in (3.12) by setting $A = 0$ or $C = 0$.

Conclusion. The autonomy of the system and the resolution of the equation $D_2(A, B) = 0$ imply that (3.12) are the only possible forms of (X_0, Y_0).

We have now to equate to zero, for each case of (3.12), the five others determinants of (3.4).

$$\text{First case} \begin{cases} X_0 = A\cos\theta \\ Y_0 = C\cos\theta \end{cases} \qquad \text{(here we have } B = D = 0\text{).}$$

In this case all the determinants of (3.4) are null without any condition on A and C. We then go to the step $j = 3$.

Step $j = 3$. After some calculations we obtain

$$a_{31} = b_{31} = c_{31} = d_{31} = 0. \tag{3.13}$$

It follows that the six determinants of this step are null without any condition on A and C. We then go to the step $j = 4$.

Step $j = 4$. We obtain after some calculations

$$
\begin{aligned}
a_{41} &= 2A(a_{30} - \tfrac{1}{6}a_{32}) - 2C(c_{30} - \tfrac{1}{6}c_{32}) + \tfrac{1}{24}(a_{12}a_{23} - c_{12}c_{23}) \\
c_{41} &= -2A(c_{30} - \tfrac{1}{6}c_{32}) - 2C(a_{30} - \tfrac{1}{6}a_{32}) - \tfrac{1}{24}(a_{12}c_{23} + a_{23}c_{12}) \\
b_{41} &= d_{41} = 0
\end{aligned}
\tag{3.14}
$$

with

$$
\begin{aligned}
&a_{30} = \tfrac{19}{72}(A^4 + C^4 - 6A^2C^2) \qquad c_{30} = \tfrac{19}{18}AC(A^2 - C^2) \\
&a_{32} = \tfrac{1}{144}(59A^4 - 107C^4 + 144A^2C^2) \qquad c_{32} = -\tfrac{1}{72}AC(131A^2 + 35C^2) \\
&a_{34} = \tfrac{1}{144}(5A^4 - C^4 - 12A^2C^2) \qquad c_{32} = \tfrac{1}{72}AC(A^2 - 7C^2)
\end{aligned}
\tag{3.15}
$$

and a_{12}, c_{12} are given in (2.18).

We can remark here that all the determinants D_4 are null except $D_4(A, C)$. The calculation of this determinant leads to

$$D_4(A, C) = \tfrac{7}{6}AC(3A^4 - 10A^2C^2 + 3C^4) \tag{3.16}$$

which vanish if

$$A = 0 \qquad \text{or} \qquad C = 0 \qquad \text{or} \qquad 3A^4 - 10A^2C^2 + 3C^4 = 0 \tag{3.17}$$

or equivalently if

$$A = 0 \qquad \text{or} \qquad C = 0 \qquad \text{or} \qquad C = \pm\sqrt{3}A \qquad \text{or} \qquad C = \pm\frac{A}{\sqrt{3}}. \tag{3.18}$$

We conclude that the only possible forms of this first case are

$$
\begin{aligned}
&X_0 = 0 \qquad X_0 = A\cos\theta \qquad X_0 = A\cos\theta \qquad X_0 = A\cos\theta \\
&Y_0 = C\cos\theta \qquad Y_0 = 0 \qquad Y_0 = \pm\sqrt{3}A\cos\theta \qquad Y_0 = \pm\frac{A}{\sqrt{3}}\cos\theta
\end{aligned}
\tag{3.19}
$$

$$\text{second case} \begin{cases} X_0 = A\cos\theta \\ Y_0 = D\sin\theta \end{cases} \qquad \text{(we have here } B = C = 0).$$

We remark here that the six determinants D_2 of the step 2 are null except the determinant

$$D_2(A, D) = \tfrac{7}{3}AD(A^2 - D^2) \tag{3.20}$$

which is null if

$$A = 0 \qquad \text{or} \qquad D = 0 \qquad \text{or} \qquad D = \pm A. \tag{3.21}$$

We deduce that the only possible forms of this second case are

$$
\begin{aligned}
&X_0 = 0 \qquad X_0 = A\cos\theta \qquad X_0 = A\cos\theta \\
&Y_0 = D\sin\theta \qquad Y_0 = 0 \qquad Y_0 = \pm A\sin\theta.
\end{aligned}
\tag{3.22}
$$

General conclusion. The algorithm can be started with one of the following eight possible cases

$$
\begin{aligned}
&X_0 = 0 \qquad X_0 = A\cos\theta \qquad X_0 = A\cos\theta \qquad X_0 = A\cos\theta \qquad X_0 = A\cos\theta \\
&Y_0 = C\cos\theta \qquad Y_0 = 0 \qquad Y_0 = \pm\sqrt{3}A\cos\theta \qquad Y_0 = \pm\frac{A}{\sqrt{3}}\cos\theta \\
&Y_0 = \pm A\sin\theta.
\end{aligned}
\tag{3.23}
$$

Actually, each case corresponds to a periodic family parametrized by the zeroth-order amplitude A. We conclude then that the LP method permits the determination of eight main

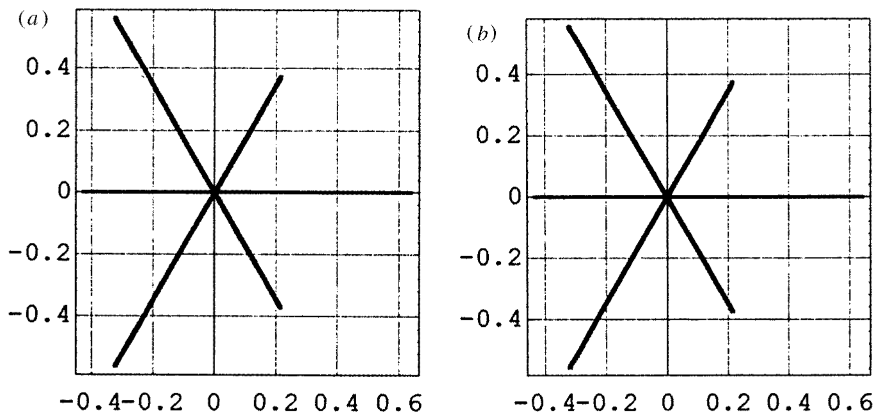

Figure 1. (*a*) Three rectilinear periodic orbits computed by the LP method. (*b*) Three rectilinear periodic orbits computed by numerical integration.

periodic families for the Hénon–Heiles system. Taking into account the symmetry and the invariance by rotations of angles $\pm\frac{2\pi}{3}$ of the potential, we can in fact distinguish only three main periodic families: the 'rectilinear-$\mathcal{R}$', the 'curvilinear-$\mathcal{V}$' and the 'circular-$\mathcal{C}$'.

Below we give the series, truncated to $O(A^{21})$, representing these three families and their periods.

3.2. *Computation by LP and test by numerical integration of the main periodic families*

As mentioned in the introduction, we can set $\epsilon = 1$ and we consequently parametrize the families by the parameter A or by the energy E (constant of motion) which is related to A by the relation

$$E = \sum_{j=1}^{\infty} E_j A^{2j}. \tag{3.24}$$

By means of the computer algebra system 'Mathematica', we obtain with exact calculation the three main periodic families as well as their periods to $O(A^{21})$.

In figures 1–3 we compare, for the energy $E = \frac{1}{8}$, the periodic orbits computed by the LP method, with those obtained by numerical integration.

(1) 'Rectilinear' periodic family $\mathcal{R}$. This family contains in fact three straight periodic families: the 'horizontal' $\mathcal{H}(y = 0)$ and two 'oblique' $\mathcal{O}^{\pm}(y = \pm\sqrt{3}x)$.

Below we give (3.25) the series representing the 'horizontal' family; the two oblique can be deduced from the previous family by rotations of angles $\pm\frac{2\pi}{3}$.

$$\begin{aligned} X(\theta) = A\cos(\theta) &+ A^2\left(\frac{1}{2} - \frac{\cos(2\theta)}{6}\right) + A^3\frac{\cos(3\theta)}{48} \\ &+A^4\left(\frac{19}{79} - \frac{59\cos(2\theta)}{432} - \frac{\cos(4\theta)}{432}\right) + A^5\left(\frac{79\cos(3\theta)}{2304} + \frac{5\cos(5\theta)}{20\,736}\right) \\ &+A^6\left(\frac{11\,897}{41\,472} - \frac{1207\cos(2\theta)}{6912} - \frac{89\cos(4\theta)}{15\,552} - \frac{\cos(6\theta)}{41\,472}\right) \end{aligned}$$

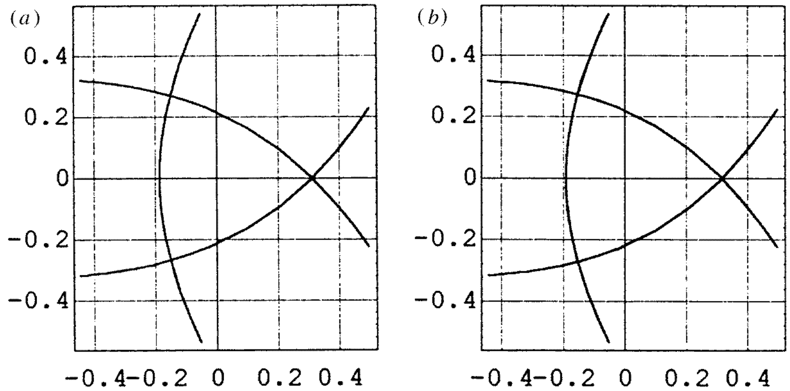

Figure 2. (*a*) Three curvilinear periodic orbits computed by the LP method. (*b*) Three curvilinear periodic orbits computed by numerical integration. (Energy $E = \frac{1}{8}$.)

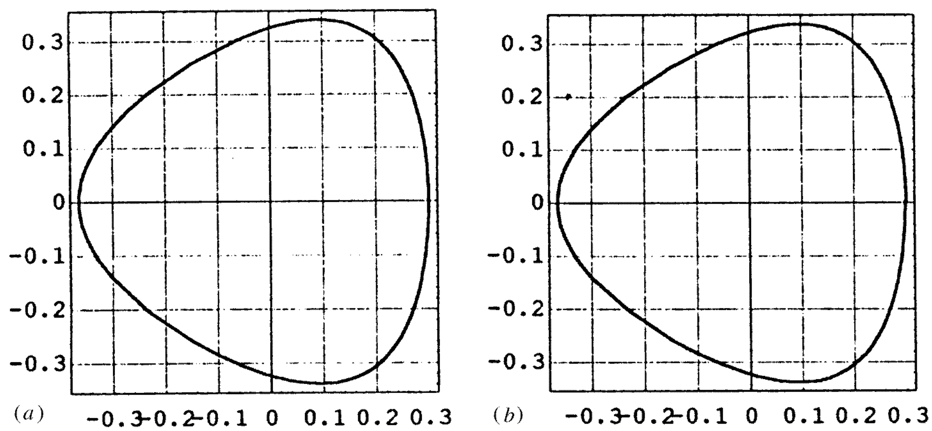

Figure 3. (*a*) Circular periodic orbit computed by the LP method. (*b*) Circular periodic orbit computed by numerical integration. (Energy $E = \frac{1}{8}$.)

$$
\begin{aligned}
&+A^7\left(\frac{19\,283\cos(3\theta)}{331\,776}+\frac{2375\cos(5\theta)}{2985\,984}+\frac{7\cos(7\theta)}{2985\,984}\right)\\
&+A^8\left(\frac{1181\,413}{2985\,984}-\frac{1151\,545\cos(2\theta)}{4478\,976}-\frac{54\,067\cos(4\theta)}{4478\,976}\right.\\
&\left.-\frac{11\cos(6\theta)}{110\,592}-\frac{\cos(8\theta)}{4478\,976}\right)\\
&+A^9\left(\frac{537\,439\cos(3\theta)}{5308\,416}+\frac{215\,725\cos(5\theta)}{107\,495\,424}+\frac{4991\cos(7\theta)}{429\,981\,696}+\frac{\cos(9\theta)}{47\,775\,744}\right)\\
&+A^{10}\left(\frac{132\,341\,179}{214\,990\,848}-\frac{528\,929\,135\cos(2\theta)}{1289\,945\,088}-\frac{1305\,943\cos(4\theta)}{53\,747\,712}\right.\\
&\left.-\frac{83\,729\cos(6\theta)}{286\,654\,464}-\frac{13\cos(8\theta)}{10\,077\,696}\right)+\cdots
\end{aligned}
\tag{3.25}
$$

$$+A^{21}\left(\frac{1846\,201\,729\,551\,096\,763\,039\cos(3\theta)}{425\,973\,332\,494\,163\,902\,464}+\cdots+\frac{7\cos(21\theta)}{1277\,919\,997\,482\,491\,707\,392}\right)+\mathrm{O}(A^{21})$$

$$Y(\theta)=0$$

$$\omega^2=1-\tfrac{5}{6}A^2-\frac{335}{864}A^4-\frac{16195}{41472}A^6-\frac{9244\,585}{17\,915\,904}A^8-\frac{505\,885\,685}{644\,972\,544}A^{10}-\frac{53\,471\,731\,165}{41\,278\,242\,816}A^{12}-\frac{120\,752\,481\,301\,295}{53\,496\,602\,689\,536}A^{14}-\frac{31\,442\,429\,175\,813\,815}{7703\,510\,787\,293\,184}A^{16}-\frac{8411\,829\,860\,794\,238\,695}{1109\,305\,553\,370\,218\,496}A^{18}-\frac{1148\,709\,686\,193\,583\,415\,155}{798\,699\,998\,426\,555\,731\,712}A^{20}+\mathrm{O}(A^{21}).$$

(2) 'Curvilinear' periodic family $\mathcal{V}$. Actually, this family contains three curvilinear families $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{V}_3$. Below we give (3.26) the series representing the family $\mathcal{V}_1$; the families $\mathcal{V}_2$ and $\mathcal{V}_3$ can be deduced from $\mathcal{V}_1$ by rotations of angles $\pm\frac{2\pi}{3}$.

$$X(\theta)=A^2\left(-\frac{1}{2}+\frac{\cos(2\theta)}{6}\right)+A^4\left(\frac{19}{72}+\frac{107\cos(2\theta)}{432}+\frac{\cos(4\theta)}{2160}\right)+A^6\left(-\frac{9241}{41\,472}+\frac{17\,761\cos(2\theta)}{10\,3680}+\frac{349\cos(4\theta)}{388\,800}+\frac{\cos(6\theta)}{115\,200}\right)+A^8\left(\frac{26\,178\,763}{74\,649\,600}+\frac{4912\,669\cos(2\theta)}{111\,974\,400}-\frac{60\,097\cos(4\theta)}{111\,974\,400}+\frac{409\cos(6\theta)}{22\,394\,880}+\frac{\cos(8\theta)}{22\,394\,880}\right)+A^{10}\left(-\frac{11\,303\,303\,447}{26\,873\,856\,000}-\frac{36\,925\,058\,981\cos(2\theta)}{161\,243\,136\,000}-\frac{26\,246\,317\cos(4\theta)}{6718\,464\,000}+\frac{463\,349\cos(6\theta)}{35\,831\,808\,000}+\frac{137\cos(8\theta)}{1182\,449\,664\,000}\right)+\cdots+A^{20}\left(\frac{140\,898\,252\,923\,082\,652\,738\,578\,947}{27\,963\,744\,157\,874\,257\,920\,000\,000}+\cdots+\frac{14\,745\,542\,820\,149\cos(20\theta)}{13\,312\,230\,645\,358\,437\,943\,046\,307\,840\,000\,000}\right)+\mathrm{O}(A^{21})$$

$$Y(\theta)=A\cos(\theta)+A^3\frac{\cos(3\theta)}{48}+A^5\left(\frac{1657\cos(3\theta)}{34\,560}+\frac{17\cos(5\theta)}{103\,680}\right)+A^7\left(\frac{120\,553\cos(3\theta)}{2764\,800}+\frac{53\,839\cos(5\theta)}{74\,649\,600}+\frac{71\cos(7\theta)}{74\,649\,600}\right)+A^9\left(\frac{24\,987\,797\cos(3\theta)}{3583\,180\,800}+\frac{3742\,409\cos(5\theta)}{2687\,385\,600}+\frac{64\,711\cos(7\theta)}{10\,749\,542\,400}+\frac{103\cos(9\theta)}{17\,915\,904\,000}\right)+\cdots+A^{21}\left(\frac{63\,416\,721\,855\,039\,974\,586\,904\,819\,141\cos(3\theta)}{157\,044\,387\,190\,621\,832\,478\,720\,000\,000}+\cdots\right. \quad (3.26)$$

$$+ \frac{542\,127\,808\,389\,263\cos(21\theta)}{3904\,920\,098\,305\,141\,796\,626\,916\,966\,400\,000\,000}\Bigg) + \mathrm{O}(A^{20})$$

$$\omega^2 = 1 - \frac{5}{6}A^2 + \frac{673}{864}A^4 - \frac{18\,053}{69\,120}A^6 + \frac{13\,759\,463}{17\,915\,904}A^8 - \frac{16\,915\,570\,541}{16\,124\,313\,600}A^{10}$$
$$+\frac{4774\,203\,682\,903}{5159\,780\,352\,000}A^{12} - \frac{87\,632\,953\,243\,730\,063}{33\,435\,376\,680\,960\,000}A^{14}$$
$$+\frac{83\,706\,128\,400\,928\,836\,749}{24\,073\,471\,210\,291\,200\,000}A^{16} - \frac{61\,194\,528\,445\,910\,688\,474\,923}{12\,710\,792\,799\,033\,753\,600\,000}A^{18}$$
$$+\frac{1772\,713\,703\,500\,289\,738\,506\,498\,277}{151\,004\,218\,452\,520\,992\,768\,000\,000}A^{20} + \mathrm{O}(A^{21}).$$

We remark here that

$$\begin{array}{l} X_{2j}(\theta) = 0 \\ Y_{2j+1}(\theta) = 0 \end{array} \qquad \forall j \in \mathbb{N}. \tag{3.27}$$

(3) 'Circular' periodic family $\mathcal{C}$. This family contains in fact two circular families $\mathcal{C}^+$ and $\mathcal{C}^-$ having the same orbit but with two opposite senses of circulation.

$$X(\theta) = A\cos(\theta) - A^2\frac{\cos(2\theta)}{3} + A^4\left(\frac{8\cos(2\theta)}{27} - \frac{\cos(4\theta)}{135}\right) + A^5\frac{\cos(5\theta)}{1620}$$
$$+A^6\left(-\frac{214\cos(2\theta)}{405} + \frac{223\cos(4\theta)}{12\,150}\right) + A^7\left(-\frac{571\cos(5\theta)}{291\,600} + \frac{\cos(7\theta)}{116\,640}\right)$$
$$+A^8\left(\frac{64\,463\,\cos(2\theta)}{54\,675} - \frac{102\,373\cos(4\theta)}{2187\,000} - \frac{\cos(8\theta)}{874\,800}\right)$$
$$+A^9\left(\frac{592\,267\cos(5\theta)}{104\,976\,000} - \frac{341\cos(7\theta)}{8398\,080}\right) + \cdots$$
$$+A^{20}\Bigg(\frac{83\,816\,790\,509\,301\,709\,072\,526\,567\cos(2\theta)}{14\,400\,884\,480\,716\,800\,000\,000}$$
$$-\frac{178\,629\,639\,940\,427\,873\,315\,207\,767\cos(4\theta)}{6336\,389\,171\,515\,392\,000\,000\,000}$$
$$-\frac{5086\,088\,610\,691\,877\,630\,528\,293\cos(8\theta)}{1328\,952\,022\,239\,161\,548\,800\,00\,000}$$
$$+\frac{3460\,611\,329\,192\,643\,548\,389\cos(10\theta)}{148\,468\,858\,734\,531\,329\,280\,000}$$
$$+\frac{44\,998\,399\,011\,033\,141\,931\cos(14\theta)}{95\,885\,851\,629\,133\,813\,598\,208\,000\,000}$$
$$+\frac{106\,900\,220\,735\,506\,097\cos(16\theta)}{85\,492\,629\,949\,052\,875\,760\,640\,000\,000}$$
$$+\frac{232\,217\cos(20\theta)}{151\,173\,284\,836\,324\,608\,000\,000}\Bigg) + \mathrm{O}(A^{21})$$

$$Y(\theta) = \pm\Bigg[A\sin(\theta) - A^2\frac{\sin(2\theta)}{3} + A^4\left(\frac{-8\sin(2\theta)}{27} - \frac{\sin(4\theta)}{135}\right) - A^5\frac{\sin(5\theta)}{1620}$$
$$+A^6\left(\frac{214\sin(2\theta)}{405} + \frac{223\sin(4\theta)}{12\,150}\right) + A^7\left(\frac{571\sin(5\theta)}{291\,600} + \frac{\sin(7\theta)}{116\,640}\right)$$
$$+A^8\left(\frac{-64\,463\sin(2\theta)}{54\,675} - \frac{102\,373\sin(4\theta)}{2187\,000} + \frac{\sin(8\theta)}{874\,800}\right) \tag{3.28}$$

$$
\begin{aligned}
&+A^9\left(\frac{-592\,267\sin(5\theta)}{104\,976\,000}-\frac{341\sin(7\theta)}{8398\,080}\right)+\cdots\\
&+A^{20}\left(\frac{-83\,816\,790\,509\,301\,709\,072\,526\,567\sin(2\theta)}{14\,400\,884\,480\,716\,800\,000\,000}\right.\\
&-\frac{178\,629\,639\,940\,427\,873\,315\,207\,767\sin(4\theta)}{6336\,389\,171\,515\,392\,000\,000\,000}\\
&+\frac{5086\,088\,610\,691\,877\,630\,528\,293\sin(8\theta)}{1328\,952\,022\,239\,161\,548\,800\,000\,000}\\
&+\frac{3460\,611\,329\,192\,643\,548\,389\sin(10\theta)}{148\,468\,858\,734\,531\,329\,280\,000\,000}\\
&-\frac{44\,998\,399\,011\,033\,141\,931\sin(14\theta)}{95\,885\,851\,629\,133\,813\,598\,208\,000\,000}\\
&-\frac{106\,900\,220\,735\,506\,097\sin(16\theta)}{85\,492\,629\,949\,052\,875\,760\,640\,000\,000}\\
&\left.+\ \frac{232\,217\sin(20\theta)}{151\,173\,284\,836\,324\,608\,000\,000}\right)+\mathrm{O}(A^{21})\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\omega^2 = 1 &+ \frac{2}{3}A^2-\frac{16}{27}A^4+\frac{428}{405}A^6-\frac{25\,7851}{109\,350}A^8+\frac{23\,207\,359}{3936\,600}A^{10}-\frac{12\,435\,443\,447}{787\,320\,000}A^{12}\\
&+\frac{113\,100\,657\,808\,997}{2550\,916\,800\,000}A^{14}-\frac{118\,199\,616\,016\,583\,611}{918\,330\,048\,000\,000}A^{16}\\
&+\frac{1393\,719\,688\,073\,359\,127\,771}{3636\,586\,990\,080\,000\,000}A^{18}\\
&-\frac{3352\,579\,515\,833\,898\,822\,938\,051}{2880\,176\,896\,143\,360\,000\,000}A^{20}+\mathrm{O}(A^{21}).
\end{aligned}
$$

We notice in figures 1–3 that numerical integration gives orbits which are indistinguishable to the naked eye from those obtained by the LP method. It is useful to show the figure grouping, for the energy $W = \frac{1}{8}$, the eight orbits computed by the LP method.

4. Study of the periods of the three main periodic families

4.1. Coefficients of the periods of the periodic familes

Since the energy is a constant of motion, we can determine the period of each periodic family in terms of E in the following form

$$T = 2\pi \sum_{j=0}^{\infty} c_j E^j. \tag{4.1}$$

In tables 1–3 we list, for $0 \leqslant j \leqslant 10$, the values c_j of the three periodic families $\mathcal{R}$, $\mathcal{V}$ and $\mathcal{C}$.

4.2. Exact rectilinear families and their periods

Recall that we have found, by means of the LP method, one main 'rectilinear' family which in fact contains three 'rectilinear' familes, the 'horizontal' $\mathcal{H}(y = 0)$ and the two 'oblique' $\mathcal{O}^{\pm}(y = \pm\sqrt{3}x)$.

Table 1. Coefficients of the 'rectilinear' period.

j	c_j
0	1
1	$\frac{5}{6}$
2	$\frac{385}{144}$
3	$\frac{85\,085}{7776}$
4	$\frac{37\,182\,145}{746\,496}$
5	$\frac{1078\,282\,205}{4478\,976}$
6	$\frac{1169\,936\,192\,425}{967\,458\,816}$
7	$\frac{36\,220\,269\,467\,525}{5804\,752\,896}$
8	$\frac{73\,201\,164\,593\,868\,025}{2229\,025\,112\,064}$
9	$\frac{190\,103\,424\,450\,275\,260\,925}{1083\,306\,204\,463\,104}$
10	$\frac{24\,675\,424\,493\,645\,728\,868\,065}{25\,999\,348\,907\,114\,496}$

Table 2. Coefficients of the 'curvilinear' period.

j	c_j
0	1
1	$\frac{5}{6}$
2	$\frac{49}{144}$
3	$-\frac{127\,967}{38\,880}$
4	$-\frac{72\,840\,859}{3732\,480}$
5	$-\frac{4789\,708\,427}{111\,974\,400}$
6	$\frac{133\,960\,064\,084\,501}{120\,932\,352\,000}$
7	$\frac{5197\,543\,150\,445\,477}{3627\,970\,560\,000}$
8	$\frac{40\,139\,072\,283\,247\,536\,077}{6965\,703\,475\,200\,000}$
9	$\frac{78\,039\,878\,077\,739\,881\,278\,059}{37\,238\,650\,778\,419\,200\,000}$
10	$-\frac{531\,657\,760\,660\,248\,287\,261\,532\,973}{4468\,638\,093\,410\,304\,000\,000}$

We will rigorously confirm this result by a direct calculation. Actually, a rectilinear solution (x, y) can be searched in the form

$$\begin{aligned} x(t) &= A \cdot \varphi(t) \\ y(t) &= B \cdot \varphi(t). \end{aligned} \tag{4.2}$$

Substituting (4.2) in the system (1.2) we obtain

$$B = 0 \qquad \text{or} \qquad B = \sqrt{3}A \qquad \text{or} \qquad B = -\sqrt{3}A \qquad \text{or} \qquad \varphi = 0. \tag{4.3}$$

The case $\varphi = 0$ corresponds to a trivial solution $(0, 0)$. The other cases represent the rectilinear families

$$y = 0 \qquad y = \sqrt{3}x \qquad y = -\sqrt{3}x. \tag{4.4}$$

We shall now search the exact periods of these rectilinear families.

The Hamiltonian of the 'horizontal' family is given by

$$H = \tfrac{1}{2}(\dot{x}^2 + x^2) - \tfrac{1}{3}x^3. \tag{4.5}$$

Table 3. Coefficients of the 'circular' period.

j	c_j
0	1
1	$-\frac{1}{3}$
2	$\frac{7}{9}$
3	$-\frac{6559}{2430}$
4	$\frac{324\,877}{29\,160}$
5	$-\frac{110\,794\,621}{2187\,000}$
6	$\frac{1157\,380\,170\,569}{4723\,920\,000}$
7	$-\frac{87\,616\,855\,970\,353}{70\,858\,800\,000}$
8	$\frac{437\,969\,924\,611\,388\,303}{68\,024\,448\,000\,000}$
9	$-\frac{24\,971\,859\,775\,321\,935\,749\,203}{7273\,173\,980\,160\,000\,000}$
10	$\frac{162\,836\,728\,825\,368\,274\,866\,249\,487}{872\,780\,877\,619\,200\,000\,000}$

The Hamiltonian of the 'oblique' families is given by

$$\frac{H}{4} = \tfrac{1}{2}(\dot{x}^2 + x^2) + \tfrac{2}{3}x^3. \tag{4.6}$$

The system is thus reduced to one degree of freedom and we can then apply a method based on the distributions [8, 9] to prove that the period of the three rectilinear families is rigorously equal to a Gauss hypergeometric series

$$\frac{T}{2\pi} = {}_2F_1(\tfrac{1}{6}, \tfrac{5}{6}; 1; 6E) = \sum_{j=0}^{\infty} d_j E^j \tag{4.7}$$

where the coefficients d_j are given by

$$d_j = \frac{(\frac{1}{6})_j \cdot (\frac{5}{6})_j \cdot 6^j}{(j/!)^2} \tag{4.8}$$

with

$$\begin{aligned} &(a)_0 = 1 \\ &(a)_j = a(a+1)\ldots(a+j-1) \qquad \text{if } j \geqslant 1. \end{aligned} \tag{4.9}$$

The comparison of the coefficients c_j (table 1) with the above hypergeometric coefficients d_j gives a perfect agreement. This therefore constitutes a check of the rectilinear periodic families obtained by the LP method.

4.3. Discussion of the convergence of the periods

We can easily prove that the radius of convergence of the hypergeometric series representing the rectilinear period is equal to 1.

Concerning the circular period, we have applied the Burlirsh–Stoer algorithm [12, 13] to the ratios sequence $(\frac{c_j}{c_{j+1}})$, where the coefficients c_j are given in the table 3. This algorithm consists of accelerating the convergence of the ratios sequence by means of a new sequence (b_j) such that $\lim_{j\to\infty} b_j = \lim_{j\to\infty} \frac{c_j}{c_{j+1}} = -\frac{1}{R}$. In table 4 we list the values $\frac{c_j}{c_{j+1}}$ and b_j for $0 \leqslant j \leqslant 24$.

Table 4.

j	$\frac{c_j}{c_{j+1}}$	b_j
0	−3	0
1	−0.428 571 428 571	−3
2	−0.288 153 681 963	−0.197 418 393 587
3	−0.242 270 151 472	−0.110 650 287 026
4	−0.219 918 392 969	−0.153 161 118 297
5	−0.206 774 219 435	−0.158 798 755 784
6	−0.198 143 409 350	−0.157 248 276 011
7	−0.192 050 131 766	−0.156 883 596 441
8	−0.187 522 053 866	−0.156 987 228 828
9	−0.184 026 244 856	−0.156 985 919 230
10	−0.181 246 586 079	−0.156 987 501 421
11	−0.178 983 861 354	−0.156 976 031 160
12	−0.177 106 364 807	−0.156 976 581 339
13	−0.175 523 571 818	−0.156 976 506 560
14	−0.174 171 227 949	−0.156 976 428 099
15	−0.173 002 473 328	−0.156 976 477 272
16	−0.171 982 334 723	−0.156 976 484 373
17	−0.171 084 184 281	−0.156 976 485 820
18	−0.170 287 392 204	−0.156 976 486 047
19	−0.169 575 729 027	−0.156 976 485 403
20	−0.168 936 252 471	−0.156 976 486 386
21	−0.168 358 515 761	−0.156 976 486 276
22	−0.167 833 994 111	−0.156 976 486 277
23	−0.167 355 662 362	−0.156 976 486 275
24	−0.166 917 679 256	−0.156 976 486 279

We can deduce that the sequence (b_j) converges (numerically) to the value $-\frac{1}{R} \approx -0.156\,976\,486$, and consequently we obtain, with a good accuracy, the radius of convergence of the circular period

$$R \approx 6.370\,380\,83. \tag{4.10}$$

On the other hand, the application of the Burlirsh–Stoer algorithm and other algorithms† (cf [13]) to the curvilinear period does not give any information about its radius of convergence. We still study this problem as well as the question of the nature of the series representing the circular and the curvilinear periods.

5. Comparison with the 'geometrical' method of Churchill–Pecelli–Rod [9]

The three authors of [9] have used a 'geometrical' method which consists of constructing the periodic orbits by exploiting the symmetries of the potential. Below we give the figure of the eight periodic orbits 'geometrically' constructed by these authors for an energy E less then the escape energy $\frac{1}{6}$.

Comparison of the above figure with figure 4 leads to the following remarks.

(1) We obtain eight orbits by the two methods.

(2) Figures 4 and 5 present three rectilinear families $y = 0$, $y = \pm\sqrt{3}x$.

(3) There is a good resemblance (forms and positions) between the 'curvilinear' orbits of figure 4 and those of figure 5.

† Epsilon-algorithm, Padé approximants.

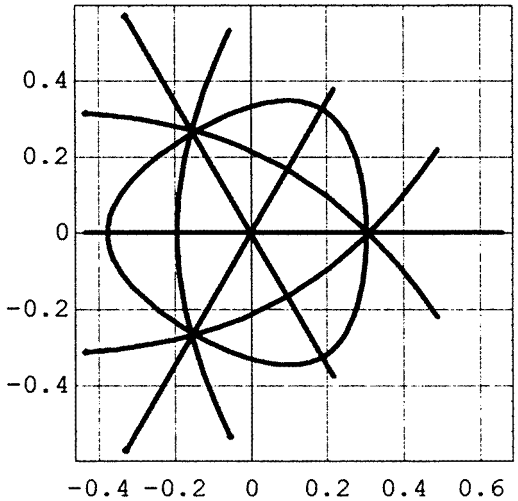

Figure 4. Eight periodic orbits computed by the LP method. (Energy $E = \frac{1}{8}$.)

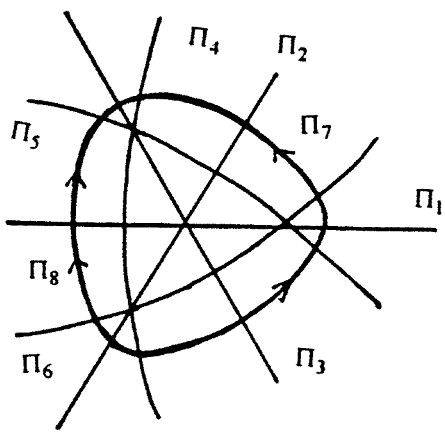

Figure 5. Eight periodic orbits 'geometrically' constructed by Churchill–Pecelli–Rod ($\epsilon = 1, E < \frac{1}{6}$).

(4) Figures 4 and 5 present two 'circular' periodic orbits.

However, we notice a disagreement in the position of the circular orbits in figures 4 and 5.

It is useful to mention that the three authors of [9] have conjectured the position of these circular orbits, not by using their 'geometrical' approach, but with 'numerical' explorations on which they have expressed some doubts. In our work, we have resolved this problem by giving not only the position of the eight periodic orbits at any time and for any sufficiently small energy E, but also their periods in terms of the energy E.

6. Conclusion

The main aim of this work is to show the interest of the LP method in the research of the periodic solutions of the Hamiltonian systems. Its importance lies in the fact that it can simultaneously be used as a means of enumeration of the main periodic families and also as a tool for the determination of these as well as their periods in the form of powers series.

In this paper, we have successfully applied the LP method to the Hénon–Heiles nonintegrable Hamiltonian system. We have proven that this system admits three main periodic families in the neighbourhood of the origin: the 'rectilinear' $\mathcal{R}$, the curvilinear $\mathcal{V}$ and the 'circular' $\mathcal{C}$. We have also shown that the period of the 'rectilinear' family is rigorously equal to a Gauss hypergeometric series (see equation (4.7)).

By means of the computer algebra system 'Mathematica', we have computed with exact accuracy the periodic families as well as their periods to higher order $O(A^{21})$, where A is a zeroth-order amplitude. We have also proven that the technique known as 'elimination of secular terms', on which the LP method is based, is mathematically equivalent to the 'alternative of Fredholm'. We have therefore tested the LP series using a numerical integration; the rectilinear families have also been checked by direct calculation. Finally we have compared our results with those of the 'geometrical' method of Churchill–Pecelli–Rod [9]. This comparison has led to a good agreement, but we have noticed a disagreement concerning the 'circular' family.

Let us mention that we have applied the LP method, in another work, to two nonintegrable Hamiltonian systems stemming from astronomy:

(I) $H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + x^2 + y^2)\epsilon x y^2$ the Barbanis–Contopoulos system.

(II) $H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + x^2 + y^2) + \epsilon x^2 y^2$ the Ollongren system.

We have found that system (I) admits, in the neighbourhood of the origin, six main periodic families from which three are rectilinear ($y = 0$ and $y = \pm\sqrt{2}x$). We have also shown that the period of the 'horizontal' family $y = 0$ is equal to 2π and the period of the two oblique $y = \pm\sqrt{2}x$ is equal to a Gauss hypergeometric series $T = 2\pi {}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; 8\epsilon^2 E)$.

Concerning system (II), we have found that it admits, in the neighbourhood of the origin, six main periodic families from which four are rectilinear ($x = 0$, $y = 0$ and $y = \pm x$). We have also shown that the period of the horizontal and vertical families ($y = 0$, $x = 0$) is equal to 2π, whereas the period of the two oblique families is equal to a Gauss hypergeometric series $T = 2\pi F_1(\frac{1}{4}, \frac{3}{4}; 1; -4\epsilon E)$.

We finally emphasize that the LP method remains little exploited in other fields such as the theory of stability and bifurcation, and the research of quasiperiodic solutions of Hamiltonian systems.

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